A Theory of Laminated Cylindrical Shells Consisting of Layers of Orthotropic Laminae

TEH-MIN HSU*
Lockheed-Georgia Company, Marietta, Ga.

AND

JAMES TING-SHUN WANG†
Georgia Institute of Technology, Atlanta, Ga.

The development of a general linear theory and the derivation of equations of motion for the analysis of laminated cylindrical shells consisting of layers of orthotropic laminae is presented. The classical Kirchhoff hypothesis of nondeformable normals commonly used for isotropic shells is abandoned so that compatible shear stresses and deformation between layers can be maintained. Transverse inextensibility of each layer is assumed; however, the normal stresses in the transverse direction are accounted for so that the peeling stresses between layers may be determined. The transverse coordinate z, when compared to the radius of the midsurface of each layered cylinder, is generally small; however, it is not neglected in the general derivation. The general procedure in the derivation is similar to that presented by Ambartsumian for orthotropic plates. However, the resulting governing differential equations are substantially more complicated than those for orthotropic plates.

Nomenclature

h^j = thickness of the jth layer of a layered cylindrical shell
N = total number of layers of laminated shell
R^j = mean radius of the jth layer of layered cylindrical shell
u^j,v^j,w^j = displacement components in longitudinal, circumferential, and normal directions, respectively
x,θ,z = cylindrical coordinates
u_0^j,v_0^j = displacement components of the middle surface of the jth layered shell in x and y directions, respectively
X^j,Y^j,Z^j = longitudinal, circumferential and normal components of surface loads applied to the jth contact surface
$M_x^j,M_\theta^j,M_{x\theta}^j,M_{\theta x}^j$ = stress couples of the jth layered shell
$N_x^j,N_\theta^j,N_{x\theta}^j,N_{\theta x}^j,Q_x^j,Q_\theta^j$ = stress resultants of the jth layered shell
$a_{ik}^j,B_{ik}^j,\Omega^j$ = material elastic constants
C_k^j,c_k^j,D_k^j,F_k^j = constants which are functions of material elastic constants and dimensions of the shell
φ^j,ψ^j = functions which characterize the variation of the transverse shear stresses
$\sigma_x^j,\sigma_\theta^j,\sigma_z^j$ = normal stress components
$\tau_{x\theta}^j,\tau_{\theta z}^j,\tau_{xz}^j$ = shear stress components
$\epsilon_x^j,\epsilon_\theta^j,\epsilon_z^j$ = normal strain components
$\gamma_{x\theta}^j,\gamma_{\theta z}^j,\gamma_{xz}^j$ = shear strain components
ρ^j = material mass density

Introduction

THE demand for increasing structural efficiency and for the weight reduction of many advanced vehicle designs has resulted in the use of a layered shell construction. The existing general theories of anisotropic layered shells were developed on the basis of the Kirchhoff hypotheses, and analyses were made by considering all layers as one equivalent layer with effective stiffnesses.

The original investigations on the theory of anisotropic shells, carried out by Shtaerman[1] in 1924, deal with the problem of the theory of symmetrically loaded orthotropic shells of revolution. The membrane theory of anisotropic shells has been considered by Ambartsumian[2] and Dong.[3] The basic equations of the theory of orthotropic shells were derived by Mushtari[4,5] in 1938. The theory of anistropic layered cylindrical shells was first investigated by Ambartsumian[6] in 1953. Eason[7] considered radial vibration for the case of an infinitely long anisotropic cylinder. Das[8] derived Donnell-Vlasov-type equations for an orthotropic single-layered cylindrical shell according to classical thin shell theory with in-plane inertia neglected. General solutions for a simply supported shell are presented, but no numerical results are given. Mirsky[9,10] applied the Frobenius series method to solve the problem of axisymmetric vibration of infinitely long orthotropic cylinders with one layer. Kalnins[11] by defining an equivalent density of a layered shell, generalized the theory of rotationally symmetric thin elastic shells and applied the results to both isotropic and orthotropic layered cylindrical shells of infinite length. Although Ahmed[12] has considered the thick shell problem, his investigation was limited to the case of infinitely long cylinders; hence, the boundary conditions have no effect on their axisymmetric vibration. Furthermore, only plane strain was considered. Bert, Baker, and Egle investigated the free vibrations of multilayer anisotropic cylindrical shells based on the Kirchhoff hypothesis.[13] Thus, all theoretical studies for vibration of anisotropic cylinders are based upon either the Kirchhoff hypothesis or upon considering only the radial vibration of infinitely long cylinders.

As indicated by Ambartsumian,[14] the hypothesis of nondeformable normals, while acceptable for isotropic shells, is often quite unacceptable for anisotropic shells, even if the anisotropic shell is relatively thin ($h/R \ll 1$). The Kirchhoff hypothesis, i.e., the one based upon the hypothesis of nondeformable normals, is quite indifferent to relations of the type G_{k3}/E_{ki}, E_{33}/E_{ki} (G_{k3} transverse moduli of shear, E_{33} transverse modulus of elasticity, E_{ki} moduli of elasticity in the direction of middle surface), and thus it often contains substantial er-

Received October 6, 1969; revision received March 30, 1970. Presented at the XXth I.A.F. Congress, Mar del Plata, Argentina, Oct. 5–12, 1969.

* Senior Aircraft Structures Engineer, Structural Methods Development Department. Associate AIAA.
‡ Professor of Engineering Science and Mechanics. Member AIAA

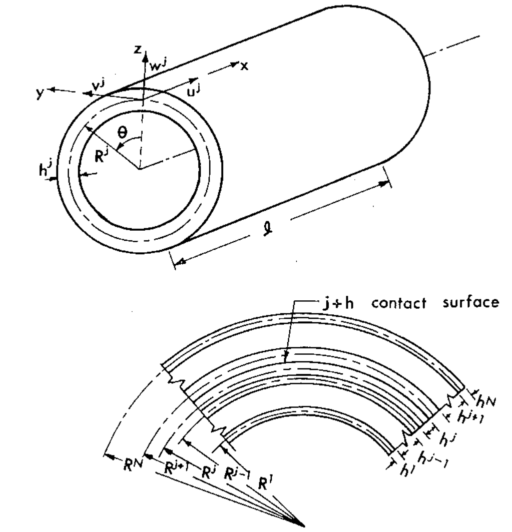

Fig. 1 Geometry of a layered cylindrical shell.

ror. Therefore, a more exact theory of shells that are constructed of anisotropic material may be developed only by abandoning the hypothesis of nondeformable normals.

In the present analysis, the hypothesis of nondeformable normals will not be invoked, but the inextensibility of the thickness together with finite length cylinders are considered; as a result, one obtains a "two-and-a-half-dimensional elasticity theory."[15]

The theory presented in this paper for a general layer may be viewed as a simplification of the Reissner-Naghdi higher-order shell theory extended to the orthotropic case, but with transverse normal strain neglected (see Ref. 17, p. 68).

It is assumed that the bond between layers of the shell is sufficiently thin so that the geometry of the shell system is not altered and the bond inertia can be neglected. In order to be able to determine interacting stresses, each layer is considered separately. Stress resultants and stress couples in terms of displacement components and shear functions are formulated, and the governing differential equations for each orthotropic cylindrical layered shell with interacting stresses acting as surface loading are derived. The general results are reduced to the rotationally symmetric deformation of N-layered cylindrical shells, and some discussions and solutions may be referred to (Hsu's work[18]).

Formulation for a General Cylindrical Layer of Orthotropic Material

The hypothesis of nondeformable normals, although acceptable for isotropic shells, is often quite unacceptable for anisotropic shells as indicated in Ref. 14, even if the anisotropic shell is relatively thin. In addition, the hypothesis will lead to incompatible interacting shear stresses between layers. It is, therefore, necessary to derive a new theory of shells without using the hypothesis of nondeformable normals.

Basic Assumptions

The basic assumptions used in this analysis are as follows.

1) All layers of the laminated shell remain elastic in the presence of deformation and obey the generalized Hooke's Law.
2) The shell is transversely inextensible.
3) No slippage takes place between layers.
4) The bond between layers is sufficiently thin that the geometry of the shell system is not altered and bond inertia can be neglected.
5) The variation of transverse shearing stresses $\tau_{xz}{}^j$ and $\tau_{\theta z}{}^j$ is represented in the following forms for a general jth layer:

$$\tau_{xz}{}^j = f_1{}^j(z)\varphi^j(x,\theta) + (z/h^j)(X^j + X^{j-1}) + (X^j - X^{j-1})/2 \quad (1a)$$

$$\tau_{yz}{}^j = f_2{}^j(z)\psi^j(x,\theta) + (z/h^j)(Y^j + Y^{j-1}) + (Y^j - Y^{j-1})/2 \quad (1b)$$

where x, y, z are cylindrical coordinates as shown in Fig. 1; X^j, X^{j-1}, Y^j, and Y^{j-1} are tangential components of the intensity of surface loads, applied to the outer surfaces of the shell; $\varphi^j(x,\theta)$, $\psi^j(x,\theta)$ are the desired functions characterizing the variation of transverse shear; $f_1{}^j(z)$ and $f_2{}^j(z)$ are functions which represent the variation of transverse shear stresses $\tau_{xz}{}^j$ and $\tau_{\theta z}{}^j$, with $f_1{}^j(\pm\frac{1}{2}h^j)$ and $f_2{}^j(\pm\frac{1}{2}h^j) = 0$, where h^j is the thickness of the jth layer.

Stress-Strain Relationships

The stress-strain relationships for an orthotropic elastic material with transverse inextensibility may be written as

$$\begin{aligned} \sigma_x{}^j &= B_{11}{}^j\epsilon_x{}^j + B_{12}{}^j\epsilon_\theta{}^j, \quad \tau_{xz}{}^j = B_{55}{}^j\gamma_{xz}{}^j \\ \sigma_\theta{}^j &= B_{12}{}^j\epsilon_x{}^j + B_{22}{}^j\epsilon_\theta{}^j, \quad \tau_{\theta z}{}^j = B_{44}{}^j\gamma_{\theta z}{}^j \\ \sigma_z{}^j &= B_{13}{}^j\epsilon_x{}^j + B_{23}{}^j\epsilon_\theta{}^j, \quad \tau_{x\theta}{}^j = B_{66}{}^j\gamma_{x\theta}{}^j \end{aligned} \quad (2)$$

or

$$\begin{aligned} \epsilon_x{}^j &= a_{11}{}^j\sigma_x{}^j + a_{12}{}^j\sigma_\theta{}^j + a_{13}{}^j\sigma_z{}^j, \quad \gamma_{\theta z}{}^j = a_{44}{}^j\tau_{\theta z}{}^j \\ \epsilon_\theta{}^j &= a_{21}{}^j\sigma_x{}^j + a_{22}{}^j\sigma_\theta{}^j + a_{23}{}^j\sigma_z{}^j, \quad \gamma_{zx}{}^j = a_{55}{}^j\tau_{zx}{}^j \\ \epsilon_z{}^j &= a_{31}{}^j\sigma_x{}^j + a_{32}{}^j\sigma_\theta{}^j + a_{33}{}^j\sigma_z{}^j = 0, \quad \gamma_{x\theta}{}^j = a_{66}{}^j\tau_{x\theta}{}^j \end{aligned} \quad (3)$$

where $\sigma_x{}^j$, $\sigma_\theta{}^j$, $\ldots$, $\tau_{x\theta}{}^j$ are stress components; $\epsilon_x{}^j$, $\epsilon_\theta{}^j$, $\ldots$, $\gamma_{x\theta}{}^j$ are strain components, and $B_{\text{in}}{}^j$ and $a_{\text{in}}{}^j$ are elastic constants. $B_{\text{in}}{}^j$ and $a_{\text{in}}{}^j$ are related as follows:

$$\begin{aligned} B_{11}{}^j &= a_{22}{}^{*j}/\Omega^j, \; B_{12}{}^j = -a_{12}{}^{*j}/\Omega^j, \; B_{22}{}^j = a_{11}{}^{*j}/\Omega^j \\ B_{44}{}^j &= 1/a_{44}{}^j, \; B_{55}{}^j = 1/a_{55}{}^j, \; B_{66}{}^j = 1/a_{66}{}^j \\ B_{13}{}^j &= -(1/a_{33}{}^j)(a_{13}{}^jB_{11}{}^j + a_{23}{}^jB_{12}{}^j) \\ B_{23}{}^j &= -(1/a_{33}{}^j)(a_{13}{}^jB_{12}{}^j + a_{23}{}^jB_{22}{}^j) \end{aligned} \quad (4)$$

where

$$\begin{aligned} a_{11}{}^{*j} &= a_{11}{}^j - [(a_{13}{}^j)^2/a_{33}{}^j], \; a_{12}{}^{*j} = a_{12}{}^j - (a_{13}{}^ja_{23}{}^j/a_{33}{}^j) \\ a_{22}{}^{*j} &= a_{22}{}^j - [(a_{23}{}^j)^2/a_{33}{}^j], \; \Omega^j = a_{11}{}^{*j}a_{22}{}^{*j} - (a_{12}{}^{*j})^2 \end{aligned}$$

and where

$$\begin{aligned} a_{11}{}^j &= 1/E_1{}^j, \; a_{12}{}^j = -\nu_{12}{}^j/E_2{}^j, \; a_{13}{}^j = -\nu_{13}{}^j/E_3{}^j \\ a_{21}{}^j &= -\nu_{21}{}^j/E_1{}^j, \; a_{22}{}^j = 1/E_2{}^j, \; a_{23}{}^j = -\nu_{23}{}^j/E_3{}^j \\ a_{31}{}^j &= -\nu_{31}{}^j/E_1{}^j, \; a_{32}{}^j = -\nu_{32}{}^j/E_2{}^j, \; a_{33}{}^j = 1/E_3{}^j \\ a_{44}{}^j &= 1/G_{23}{}^j, \; a_{55}{}^j = 1/G_{13}{}^j, \; a_{66}{}^j = 1/G_{12}{}^j \end{aligned}$$

$(E_1{}^j,E_2{}^j,E_3{}^j)$, $(G_{12}{}^j,G_{13}{}^j,G_{23}{}^j)$, and $(\nu_{12}{}^j,\nu_{13}{}^j,\nu_{23}{}^j)$ are generalized elastic and shear moduli and Poisson's ratios, respectively.

Strain-Displacement Relationships

The strain-displacement relationships for a cylindrical shell are as follows:

$$\epsilon_x{}^j = \partial u^j/\partial x, \quad \gamma_{x\theta}{}^j = [1/(R^j + z)](\partial u^j/\partial\theta) + (\partial v^j/\partial x)$$

$$\epsilon_\theta{}^j = \frac{1}{R^j + z}\left(\frac{\partial v^j}{\partial\theta} + w^j\right), \quad \gamma_{\theta z}{}^j = \frac{\partial v^j}{\partial z} - \frac{v^j}{R^j + z} + \frac{1}{R^j + z}\frac{\partial w^j}{\partial\theta} \quad (5)$$

$$\epsilon_z{}^j = \partial w^j/\partial z, \; \gamma_{xz}{}^j = (\partial u^j/\partial z) + (\partial w^j/\partial x)$$

where u^j, v^j, and w^j are displacement components in longitudinal, circumferential, and normal directions, respectively, shown in Fig. 1; and R^j is the mean radius of the jth layer.

These equations are reduced directly from the relations for general shells as in Ref. 16 and other textbooks.

Stress Resultants and Stress Couples

By the assumption of transverse inextensibility, from the third equation of Eq. (5), one obtains

$$\epsilon_z{}^j = \partial w^j/\partial z = 0$$

or

$$w^j = w(x,\theta) \qquad (6)$$

The substitution of Eqs. (1) into the 4th and 5th equations of Eq. (2) results in the expressions

$$\gamma_{xz}{}^j = a_{55}{}^j\{f_1{}^j\varphi^j + (z/h^j)(X^j + X^{j-1}) + [(X^j - X^{j-1})/2]\} \qquad (7)$$

$$\gamma_{\theta z}{}^j = a_{44}{}^j\{f_2{}^j\psi^j + (z/h^j)(Y^j + Y^{j-1}) + [(Y^j - Y^{j-1})/2]\} \qquad (8)$$

Substituting Eqs. (6–8) into the 5th and 6th equations of Eq. (5) and integrating with respect to the normal coordinate z yields the following expressions for longitudinal and circumferential displacements:

$$u^j(x,\theta,z) = u_0{}^j(x,\theta) - z[\partial w^j(x,\theta)/\partial x] + f_3{}^j(z)\varphi^j(x,\theta) + f_4{}^j(z)X^j(x,\theta) + f_5{}^j(z)X^{j-1}(x,\theta) \qquad (9)$$

and

$$v^j(x,\theta,z) = v_0{}^j(x,\theta)[1 + (z/R^j)] + f_6{}^j(z)\psi^j(x,\theta) - (z/R^j)\partial w^j(x,\theta)/\partial\theta + f_7{}^j(z)Y^j(x,\theta) + f_8{}^j(z)Y^{j-1}(x,\theta) \qquad (10)$$

where $u_0{}^j$, $v_0{}^j$ are the tangential displacements of the middle surface of the jth layer in the x and θ directions, respectively, and where

$$f_3{}^j(z) = a_{55}{}^j\int_0^z f_1{}^j(\zeta)d\zeta$$

$$f_4{}^j(z) = (a_{55}{}^j/2)\cdot z[1 + (z/h^j)]$$

$$f_5{}^j(z) = (a_{55}{}^j/2)\cdot z[-1 + (z/h^j)] \qquad (11)$$

$$f_6{}^j(z) = a_{44}{}^j(R^j + z)\int_0^z \frac{f_2{}^j(\zeta)}{R^j + \zeta}\,d\zeta$$

$$f_7{}^j(z) = (a_{44}{}^j/2)(R^j + z)\{(z/h^j) + [\tfrac{1}{2} - (R^j/h^j)]\ln[(R^j + z)/R^j]\}$$

$$f_8{}^j(z) = (a_{44}{}^j/2)(R^j + z)\{(z/h^j) - [\tfrac{1}{2} + (R^j/h^j)] \times \ln[(R^j + z)/R^j]\}$$

From Eq. (5), the strain-displacement relations then have the forms

$$\epsilon_x{}^j = \frac{\partial u_0{}^j}{\partial x} - z\frac{\partial^2 w^j}{\partial x^2} + f_3{}^j\frac{\partial\varphi^j}{\partial x} + f_4{}^j\frac{\partial X^j}{\partial x} + f_5{}^j\frac{\partial X^{j-1}}{\partial x}$$

$$\epsilon_\theta{}^j = \frac{1}{R^j}\frac{\partial v_0{}^j}{\partial\theta} + \frac{f_6{}^j}{R^j + z}\frac{\partial\psi^j}{\partial\theta} - \frac{z}{R^j(R^j + z)}\frac{\partial^2 w^j}{\partial\theta^2} + \frac{w^j}{R^j + z} + \frac{f_7{}^j}{R^j + z}\frac{\partial Y^j}{\partial\theta} + \frac{f_7{}^j}{R^j + z}\frac{\partial Y^{j-1}}{\partial\theta} \qquad (12)$$

$$\epsilon_z{}^j = \frac{\partial w^j}{\partial z} = 0$$

$$\gamma_{x\theta}{}^j = \frac{1}{R^j + z}\frac{\partial u_0{}^j}{\partial\theta} - \frac{z(2R^j + z)}{R^j(R^j + z)}\frac{\partial^2 w^j}{\partial x\partial\theta} + \frac{f_3{}^j}{R^j + z}\frac{\partial\varphi^j}{\partial\theta} + \left(1 + \frac{z}{R^j}\right)\frac{\partial v_0{}^j}{\partial x} + f_6{}^j\frac{\partial\psi^j}{\partial x} + \frac{f_4{}^j}{R^j + z}\frac{\partial X^j}{\partial\theta} + \frac{f_5{}^j}{R^j + z}\frac{\partial X^{j-1}}{\partial\theta} + f_8{}^j\frac{\partial Y^j}{\partial x} + f_8{}^j\frac{\partial Y^{j-1}}{\partial x}$$

and $\gamma_{xz}{}^j$, $\gamma_{\theta z}{}^j$ are indicated in Eqs. (7) and (8).

Substituting Eq. (12) into Eq. (2), one can express the stress components in terms of displacement components and normal coordinate.

Stress resultants $\{N_x{}^j, N_\theta{}^j, N_{x\theta}{}^j, N_{\theta x}{}^j, Q_x{}^j, Q_\theta{}^j\}$ and stress couples $\{M_x{}^j, M_\theta{}^j, M_{x\theta}{}^j, M_{\theta x}{}^j\}$ can be formulated in terms of displacement of middle surface of the shell by the integration of stresses over the shell thickness as follows:

$$\{N_x{}^j, N_{x\theta}{}^j, Q_x{}^j\} = \int_{-h^j/2}^{h^j/2}\{\sigma_x{}^j, \tau_{x\theta}{}^j, \tau_{xz}{}^j\}\left(1 + \frac{\zeta}{R^j}\right)d\zeta$$

$$\{N_\theta{}^j, N_{\theta x}{}^j, Q_\theta{}^j\} = \int_{-h^j/2}^{h^j/2}\{\sigma_\theta{}^j, \tau_{\theta x}{}^j, \tau_{\theta z}{}^j\}d\zeta \qquad (13)$$

$$\{M_x{}^j, M_{x\theta}{}^j\} = \int_{-h^j/2}^{h^j/2}\{\sigma_x{}^j, \tau_{x\theta}{}^j\}\cdot\zeta\left(1 + \frac{\zeta}{R^j}\right)d\zeta$$

and

$$\{M_\theta{}^j, M_{\theta x}{}^j\} = \int_{-h^j/2}^{h^j/2}\{\sigma_\theta{}^j, \tau_{\theta x}{}^j\}\cdot\zeta d\zeta$$

After carrying through the integrations, one obtains

$$N_x{}^j = B_{11}{}^jh^j\frac{\partial u_0{}^j}{\partial x} + B_{12}\frac{h^j}{R^j}\left(\frac{\partial v_0{}^j}{\partial\theta} + w^j\right) - \frac{B_{11}{}^j(h^j)^3}{12R^j}\frac{\partial^2 w^j}{\partial x^2} + B_{11}{}^j\left[\left(g_1{}^j + \frac{1}{R^j}g_4{}^j\right)\frac{\partial\varphi^j}{\partial x} + \left(g_2{}^j + \frac{1}{R^j}g_5{}^j\right)\frac{\partial X^j}{\partial x} + \left(g_3{}^j + \frac{1}{R^j}g_6{}^j\right)\frac{\partial X^{j-1}}{\partial x}\right] + \frac{B_{12}{}^j}{R^j}\left[g_{16}{}^j\frac{\partial\psi^j}{\partial\theta} + g_{17}{}^j\frac{\partial Y^j}{\partial\theta} + g_{18}{}^j\frac{\partial Y^{j-1}}{\partial\theta}\right] \qquad (14)$$

$$N_\theta{}^j = B_{12}{}^jh^j\frac{\partial u_0{}^j}{\partial x} + B_{22}{}^j\frac{h^j}{R^j}\frac{\partial v_0{}^j}{\partial\theta} + B_{22}{}^j\left(\frac{h^j}{R^j} + c_1{}^j\right)w^j + c_1{}^jB_{22}{}^j\frac{\partial^2 w^j}{\partial\theta^2} + B_{12}{}^j\left(g_1{}^j\frac{\partial\varphi^j}{\partial x} + g_2{}^j\frac{\partial X^j}{\partial x} + g_3{}^j\frac{\partial X^{j-1}}{\partial x}\right) + B_{22}{}^j\left(g_{25}{}^j\frac{\partial\psi^j}{\partial\theta} + g_{26}{}^j\frac{\partial Y^j}{\partial\theta} + g_{27}{}^j\frac{\partial Y^{j-1}}{\partial\theta}\right)$$

$$N_{x\theta}{}^j = \frac{B_{66}{}^j}{R^j}\left[h^j\frac{\partial u_0{}^j}{\partial\theta} + \left(R^jh^j + \frac{(h^j)^3}{12R^j}\right)\frac{\partial v_0{}^j}{\partial x} - \frac{(h^j)^3}{12R^j}\frac{\partial^2 w^j}{\partial x\partial\theta} + g_1{}^j\frac{\partial\varphi^j}{\partial\theta} + g_2{}^j\frac{\partial X^j}{\partial\theta} + g_3{}^j\frac{\partial X^{j-1}}{\partial\theta} + (R^jg_{16}{}^j + g_{19}{}^j)\frac{\partial\psi^j}{\partial x} + (R^jg_{17}{}^j + g_{20}{}^j)\frac{\partial Y^j}{\partial x} + (R^jg_{18}{}^j + g_{21}{}^j)\frac{\partial Y^{j-1}}{\partial x}\right]$$

$$N_{\theta x}{}^j = B_{66}{}^j\left[\left(c_1{}^j + \frac{h^j}{R^j}\right)\frac{\partial u_0{}^j}{\partial\theta} + c_1{}^jR^j\frac{\partial^2 w^j}{\partial x\partial\theta} + h^j\frac{\partial v_0{}^j}{\partial x} + g_{10}{}^j\frac{\partial\varphi^j}{\partial\theta} + g_{11}{}^j\frac{\partial X^j}{\partial\theta} + g_{12}{}^j\frac{\partial X^{j-1}}{\partial\theta} + g_{16}{}^j\frac{\partial\psi^j}{\partial x} + g_{17}{}^j\frac{\partial Y^j}{\partial x} + g_{18}{}^j\frac{\partial Y^{j-1}}{\partial x}\right]$$

$$Q_x{}^j = g_{31}{}^j\varphi^j + \frac{h^j}{2}\left(1 + \frac{h^j}{6R^j}\right)X^j - \frac{h^j}{2}\left(1 - \frac{h^j}{6R^j}\right)X^{j-1}$$

$$Q_\theta{}^j = g_{32}{}^j\psi^j + \frac{h^j}{2}(Y^j - Y^{j-1})$$

$$M_x{}^j = \frac{(h^j)^3}{12R^j}\left(B_{11}{}^j\frac{\partial u_0{}^j}{\partial x} + \frac{B_{12}{}^j}{R^j}\frac{\partial v_0{}^j}{\partial\theta} - B_{11}{}^jR^j\frac{\partial^2 w^j}{\partial x^2} - \frac{B_{12}{}^j}{R^j}\frac{\partial^2 w^j}{\partial\theta^2}\right) + B_{11}{}^j\left(g_7{}^j\frac{\partial\varphi^j}{\partial x} + g_8{}^j\frac{\partial X^j}{\partial x} + g_9{}^j\frac{\partial X^{j-1}}{\partial x}\right) + \frac{B_{12}{}^j}{R^j}\left(g_{19}{}^j\frac{\partial\psi^j}{\partial\theta} + g_{20}{}^j\frac{\partial Y^j}{\partial\theta} + g_{21}{}^j\frac{\partial Y^{j-1}}{\partial\theta}\right)$$

$$M_\theta{}^j = -\frac{B_{12}{}^j(h^j)^3}{12}\frac{\partial^2 w^j}{\partial x^2} - B_{22}{}^j R^j c_1{}^j\left(w^j + \frac{\partial^2 w^j}{\partial\theta^2}\right) + B_{12}{}^j\left(g_4{}^j\frac{\partial\varphi^j}{\partial x} + g_5{}^j\frac{\partial X^j}{\partial x} + g_6{}^j\frac{\partial X^{j-1}}{\partial x}\right) + B_{22}{}^j\left(g_{28}{}^j\frac{\partial\psi^j}{\partial\theta} + g_{29}{}^j\frac{\partial Y^j}{\partial\theta} + g_{30}{}^j\frac{\partial Y^{j-1}}{\partial\theta}\right)$$

$$M_{x\theta}{}^j = \frac{B_{66}{}^j}{R^j}\left[\frac{(h^j)^3}{6}\left(\frac{\partial v_0{}^j}{\partial x} - \frac{\partial^2 w^j}{\partial x\partial\theta}\right) + g_4{}^j\frac{\partial\varphi^j}{\partial\theta} + g_5{}^j\frac{\partial X^j}{\partial\theta} + g_6{}^j\frac{\partial X^{j-1}}{\partial\theta} + g_{22}{}^j\frac{\partial\psi^j}{\partial x} + g_{23}{}^j\frac{\partial Y^j}{\partial x} + g_{24}{}^j\frac{\partial Y^{j-1}}{\partial x}\right]$$

$$M_{\theta x}{}^j = B_{66}{}^j\left\{-R^j c_1{}^j\frac{\partial u_0{}^j}{\partial\theta} - \left[c_1{}^j(R^j)^2 + \frac{(h^j)^3}{12R^j}\right]\frac{\partial^2 w^j}{\partial x\partial\theta} + \frac{(h^j)^3}{12R^j}\frac{\partial v_0{}^j}{\partial x} + g_{13}{}^j\frac{\partial\varphi^j}{\partial\theta} + g_{14}{}^j\frac{\partial X^j}{\partial\theta} + g_{15}{}^j\frac{\partial X^{j-1}}{\partial\theta} + g_{19}{}^j\frac{\partial\psi^j}{\partial x} + g_{20}{}^j\frac{\partial Y^j}{\partial x} + g_{21}{}^j\frac{\partial Y^{j-1}}{\partial x}\right\}$$

where

$$c_1{}^j = \tfrac{1}{12}(h^{j3}/R^j) + \tfrac{1}{80}(h^j/R^j)^5$$

$$\{g_1{}^j,g_2{}^j,g_3{}^j\} = \int_{-h/2}^{h/2}\{f_3{}^j,f_4{}^j,f_5{}^j\}dz$$

$$\{g_4{}^j,g_5{}^j,g_6{}^j\} = \int_{-h/2}^{h/2}\{f_3{}^j,f_4{}^j,f_5{}^j\}\cdot z dz$$

$$\{g_7{}^j,g_8{}^j,g_9{}^j\} = \int_{-h/2}^{h/2}\{f_3{}^j,f_4{}^j,f_5{}^j\}\cdot z\left(1 + \frac{z}{R^j}\right)dz$$

$$\{g_{10}{}^j,g_{11}{}^j,g_{12}{}^j\} = \int_{-h/2}^{h/2}\{f_3{}^j,f_4{}^j,f_5{}^j,\}\cdot\frac{1}{R^j + z}dz \quad (15)$$

$$\{g_{13}{}^j,g_{14}{}^j,g_{15}{}^j\} = \int_{-h/2}^{h/2}\{f_3{}^j,f_4{}^j,f_5{}^j\}\cdot\frac{z}{R^j + z}dz = \{g_1 - R^j g_{10}{}^j, g_2{}^j - R^j g_{11}{}^j, g_3{}^j - R^j g_{12}{}^j\}$$

$$\{g_{16}{}^j,g_{17}{}^j,g_{18}{}^j\} = \int_{-h/2}^{h/2}\{f_6{}^j,f_7{}^j,f_8{}^j\}dz$$

$$\{g_{19}{}^j,g_{20}{}^j,g_{21}{}^j\} = \int_{-h/2}^{h/2}\{f_6{}^j,f_7{}^j,f_8{}^j\}\cdot z dz$$

$$\{g_{22}{}^j,g_{23}{}^j,g_{24}{}^j\} = \int_{-h/2}^{h/2}\{f_6{}^j,f_7{}^j,f_8{}^j\}\cdot z(R^j + z)dz$$

$$\{g_{25}{}^j,g_{26}{}^j,g_{27}{}^j\} = \int_{-h/2}^{h/2}\{f_6,{}^j f_7{}^j,f_8{}^j\}\cdot\frac{1}{R^j + z}dz$$

$$\{g_{28}{}^j,g_{29}{}^j,g_{30}{}^j\} = \int_{-h/2}^{h/2}\{f_6{}^j,f_7{}^j,f_8{}^j\}\cdot\frac{z}{R^j + z}dz$$

$$g_{31}{}^j = \int_{-h/2}^{h/2} f_1{}^j\left(1 + \frac{z}{R^j}\right)dz$$

$$g_{32}{}^j = \int_{-h/2}^{h/2} f_2{}^j dz$$

It may be noted that the inclusion of z/R^j in Eqs. (13) leads to the relations $T_{x\phi} \neq T_{\phi x}$, and $M_{x\phi} \neq M_{\phi x}$. Note also that terms involving the logarithm through the integration of Eq. (13) are expanded in a power series with the first three terms retained in the expressions (14). This would make the equations easy to compare with other theories.

Governing Differential Equations of Motion

The equations of motion of a general shell element can be obtained by including the inertia terms in the equilibrium equations that are available in any standard textbook on shell theories, such as Ref. 16. When the coordinate system coincides with the lines of principal curvature, the equations of motion of the jth layer of layered cylindrical shell become

$$(\partial N_x{}^j/\partial x) + (1/R^j)(\partial N_{\theta}{}^j{}_x/\partial\theta) + q_x{}^j = \rho^j h^j(\partial^2 u_0{}^j/\partial t^2) \quad (16)$$

$$(\partial N_{x\theta}{}^j/\partial x) + (1/R^j)(\partial N_\theta{}^j/\partial\theta) + (Q_\theta{}^j/R^j) + q_\theta{}^j = \rho^j h^j(\partial^2 v_0{}^j/\partial t^2)$$

$$(\partial Q_x{}^j/\partial x) + (1/R^j)(\partial Q_\theta{}^j/\partial\theta) - (N_\theta{}^j/R^j) + q_n{}^j = \rho^j h^j(\partial^2 w^j/\partial t^2)$$

$$(\partial M_x{}^j/\partial x) + (1/R^j)(\partial M_{\theta x}{}^j/\partial\theta) - Q_x{}^j = -(\rho^j(h^j)^3/12(\partial^3 w^j/\partial x\partial t^2)$$

$$(\partial M_{x\theta}{}^j/\partial x) + (1/R^j)(\partial M_\theta{}^j/\partial\theta) - Q_\theta{}^j = -[\rho^j(h^j)^3/12](\partial^3 w^j/\partial\theta\partial t^2)$$

$$N_{x\theta}{}^j - N_{\theta x}{}^j - (M_{\theta x}{}^j/R^j) = 0$$

where $q_x{}^j$, $q_\theta{}^j$ and $q_n{}^j$ are surface loading functions.

Substitution of the set of Eqs. (14) into the equations of motion (16) yields the following governing differential equations of motion for a general orthotropic jth layer of a layered cylindrical shell:

$$B_{11}{}^j h^j\frac{\partial^2 u_0{}^j}{\partial x^2} + \frac{B_{66}{}^j}{R^j}\left(\frac{h^j}{R^j} + c_1{}^j\right)\frac{\partial^2 u_0{}^j}{\partial\theta^2} + (B_{12}{}^j + B_{66}{}^j)\frac{h^j}{R^j}\frac{\partial^2 v_0{}^j}{\partial x\partial\theta} + B_{12}{}^j\frac{h^j}{R^j}\frac{\partial w^j}{\partial x} - \frac{B_{11}{}^j(h^j)^3}{12R^j}\frac{\partial^3 w^j}{\partial x^3} + c_1{}^j B_{66}{}^j\frac{\partial^3 w^j}{\partial x\partial\theta^2} + B_{11}{}^j\left[\left(g_1{}^j + \frac{1}{R^j}g_4{}^j\right)\frac{\partial^2\varphi^j}{\partial x^2} + \left(g_2{}^j + \frac{1}{R^j}g_5{}^j\right)\frac{\partial^2 X^j}{\partial x^2} + \left(g_3{}^j + \frac{1}{R^j}g_6{}^j\right)\frac{\partial^2 X^{j-1}}{\partial x^2}\right] + \frac{B_{66}{}^j}{R^j}\left[g_{10}{}^j\frac{\partial^2\varphi^j}{\partial\theta^2} + g_{11}{}^j\frac{\partial^2 X^j}{\partial\theta^2} + g_{12}{}^j\frac{\partial^2 X^{j-1}}{\partial\theta^2}\right] + \frac{B_{12}{}^j + B_{66}{}^j}{R^j}\left(g_{16}{}^j\frac{\partial^2\psi^j}{\partial x\partial\theta} + g_{17}{}^j\frac{\partial^2 Y^j}{\partial x\partial\theta} + g_{18}{}^j\frac{\partial^2 Y^{j-1}}{\partial x\partial\theta}\right) + q_x{}^j = \rho^j h^j\frac{\partial^2 u_0{}^j}{\partial t^2} \quad (17)$$

$$(B_{12}{}^j + B_{66}{}^j)\frac{h^j}{R^j}\frac{\partial^2 u_0{}^j}{\partial x\partial\theta} + \frac{B_{66}{}^j}{R^j}\left[R^j h^j + \frac{(h^j)^3}{12R^j}\right]\frac{\partial^2 v_0{}^j}{\partial x^2} + B_{22}{}^j\frac{h^j}{(R^j)^2}\frac{\partial^2 v_0{}^j}{\partial\theta^2} + \frac{B_{22}{}^j}{R^j}\left(\frac{h^j}{R^j} + c_1{}^j\right)\frac{\partial w^j}{\partial\theta} - \frac{B_{66}{}^j}{R^j}\frac{(h^j)^3}{12R^j}\frac{\partial^3 w^j}{\partial x^2\partial\theta} + \frac{B_{22}{}^j}{R^j}c_1{}^j\frac{\partial^3 w^j}{\partial\theta^3} + \frac{(B_{12}{}^j + B_{66}{}^j)}{R^j}\times\left[g_1{}^j\frac{\partial^2\varphi^j}{\partial x\partial\theta} + g_2{}^j\frac{\partial^2 X^j}{\partial x\partial\theta} + g_3{}^j\frac{\partial^2 X^{j-1}}{\partial x\partial\theta}\right] + g_{32}{}^j\frac{\psi^j}{R^j} + \frac{B_{66}{}^j}{R^j}\left[(R^j g_{16}{}^j + g_{19}{}^j)\frac{\partial^2\psi^j}{\partial x^2} + (R^j g_{17}{}^j + g_{20}{}^j)\frac{\partial^2 Y^j}{\partial x^2} + (R^j g_{18}{}^j + g_{21}{}^j)\frac{\partial^2 Y^{j-1}}{\partial x^2}\right] + \frac{B_{22}{}^j}{R^j}\left[g_{25}{}^j\frac{\partial^2\psi^j}{\partial\theta^2} + g_{26}{}^j\frac{\partial^2 Y^j}{\partial\theta^2} + g_{27}{}^j\frac{\partial^2 Y^{j-1}}{\partial\theta^2}\right] + \frac{h^j}{2R^j}(Y^j - Y^{j-1}) + q_\theta{}^j = \rho^j h^j\frac{\partial^2 v_0{}^j}{\partial t^2}$$

$$B_{12}^j \frac{h^j}{R^j}\frac{\partial u_0^j}{\partial x} + B_{22}^j \frac{h^j}{(R^j)}\frac{\partial v_0^j}{\partial \theta} + \frac{B_{22}^j}{R^j}\left(\frac{h^j}{R^j} + c_1^j\right) w^j + \frac{B_{22}^j}{R^j} c_1^j \frac{\partial^2 w^j}{\partial \theta^2} + \left(\frac{B_{12}^j}{R^j} g_1^j - g_{31}^j\right)\frac{\partial \varphi^j}{\partial x} + \left[\frac{B_{12}^j}{R^j} g_2^j - \frac{h^j}{2}\left(1 + \frac{h^j}{6R^j}\right)\right]\frac{\partial X^j}{\partial x} + \left[\frac{B_{12}^j}{R^j} g_3^j + \frac{h^j}{2}\left(1 - \frac{h^j}{6R^j}\right)\right] \times \frac{\partial X^{j-1}}{\partial x} + \left[\frac{B_{22}^j}{R^j} g_{25}^j - \frac{1}{R^j} g_{32}^j\right]\frac{\partial \psi^j}{\partial \theta} + \left[\frac{B_{22}^j}{R^j} g_{26}^j - \frac{h^j}{2R^j}\right]\frac{\partial Y^j}{\partial \theta} + \left[\frac{B_{22}^j}{R^j} g_{27}^j + \frac{h^j}{2R^j}\right] \times \frac{\partial Y^{j-1}}{\partial \theta} - q_n^j = -\rho^j h^j \frac{\partial^2 w^j}{\partial t^2}$$

$$\frac{B_{11}^j (h^j)^3}{12R^j}\frac{\partial^2 u_0^j}{\partial x^2} - B_{66}^j c_1^j \frac{\partial^2 u_0^j}{\partial \theta^2} + \frac{(h^j)^3}{12(R^j)^2}(B_{12}^j + B_{66}^j) \times \frac{\partial^2 v_0^j}{\partial x \partial \theta} - \frac{B_{11}^j (h^j)^3}{12}\frac{\partial^3 w^j}{\partial x^3} - \left[B_{66}^j c_1^j R^j + \frac{(h^j)^3}{12(R^j)^2} \times (B_{12}^j + B_{66}^j)\right]\frac{\partial^3 w^j}{\partial x \partial \theta^2} + B_{11}^j\left(g_7^j \frac{\partial^2 \varphi^j}{\partial x^2} + g_8^j \frac{\partial^2 X^j}{\partial x^2} + g_9^j \frac{\partial^2 X^{j-1}}{\partial x^2}\right) + \frac{B_{66}^j}{R^j}\left[g_{13}^j \frac{\partial^2 \varphi^j}{\partial \theta^2} + g_{14}^j \frac{\partial^2 X^j}{\partial \theta^2} + g_{15}^j \frac{\partial^2 X^{j-1}}{\partial \theta^2}\right] + \frac{B_{12}^j + B_{66}^j}{R^j}\left(g_{19}^j \frac{\partial^2 \psi^j}{\partial x \partial \theta} + g_{20}^j \frac{\partial^2 Y^j}{\partial x \partial \theta} + g_{21}^j \frac{\partial^2 Y^{j-1}}{\partial x \partial \theta}\right) - g_{31}^j \varphi^j - \frac{h^j}{2}\left(1 + \frac{h^j}{6R^j}\right) X^j + \frac{h^j}{2}\left(1 - \frac{h^j}{6R^j}\right) X^{j-1} = -\frac{\rho^j (h^j)^3}{12}\frac{\partial^3 w^j}{\partial x \partial t^2}$$

$$\frac{B_{66}^j (h^j)^3}{6R^j}\frac{\partial^2 v_0^j}{\partial x^2} - c_1^j B_{22}^j\left(\frac{\partial w^j}{\partial \theta} + \frac{\partial^3 w^j}{\partial \theta^3}\right) - \left[\frac{B_{12}^j (h^j)^3}{12R^j} + \frac{(h^j)^3}{6R^j} B_{66}^j\right]\frac{\partial^3 w^j}{\partial x^2 \partial \theta} + \frac{B_{12}^j + B_{66}^j}{R^j}\left(g_4^j \frac{\partial^2 \varphi^j}{\partial x \partial \theta} + g_5^j \frac{\partial^2 X^j}{\partial x \partial \theta} + g_6^j \frac{\partial^2 X^{j-1}}{\partial x \partial \theta}\right) - g_{32}^j \psi^j - \frac{h^j}{2}(Y^j - Y^{j-1}) + \frac{B_{66}^j}{R^j} \times \left(g_{22}^j \frac{\partial^2 \psi^j}{\partial x^2} + g_{23}^j \frac{\partial^2 Y^j}{\partial x^2} + g_{24}^j \frac{\partial^2 Y^{j-1}}{\partial x^2}\right) + \frac{B_{22}^j}{R^j} \times \left(g_{28}^j \frac{\partial^2 \psi^j}{\partial \theta^2} + g_{29}^j \frac{\partial^2 Y^j}{\partial \theta^2} + g_{30}^j \frac{\partial^2 Y^{j-1}}{\partial \theta^2}\right) = -\frac{\rho^j (h^j)^3}{12}\frac{\partial^3 w^j}{\partial \theta \partial t^2}$$

The last equation of Eq. (16) is satisfied identically.

Boundary Conditions

The boundary conditions commonly considered in practice are shown below. For convenience of discussion, only the homogeneous boundary conditions are given for the edge defined coordinate line x = constant.

1) Simply supported edge:

$$u_0^j = v_0^j = w^j = M_x^j = \partial v^j/\partial z|_{z=0} = 0 \qquad (18a)$$

If the supporting edge is without axial constraint, then the condition $u_0^j = 0$ is repaced by the condition $N_x^j = 0$.

(2) Fully fixed edge:

$$u_0^j = v_0^j = w^j = \partial w^j/\partial x = \partial v^j/\partial x|_{z=0} = 0 \qquad (18b)$$

3) Free edge:

$$N_x^j = Q_x^j = N_{x\theta}^j = M_x^j = M_{x\theta}^j = 0 \qquad (18c)$$

It should be noted that five boundary conditions per edge need to be prescribed rather than four as in thin shell theory.

Differential Equations for Rotationally Symmetric Vibrations

For the case of rotationally symmetric vibration, the functions u_0^j, w^j, φ^j, X^j and X^{j-1} are functions of the longitudinal coordinate x and time only, whereas v_0^j, ψ^j, Y^j, and Y^{j-1} are zero. Thus, the following three governing differential equations of motion for a general layered orthotropic cylindrical shell reduced from Eqs. (17a) to (17e):

$$C_1^j \frac{\partial^2 u_0}{\partial x^2} + C_2^j \frac{\partial w^j}{\partial x} + C_3^j \frac{\partial^3 w^j}{\partial x^3} + C_4^j \frac{\partial^2 \varphi^j}{\partial x^2} + C_5^j \frac{\partial^2 X^j}{\partial x^2} + C_6^j \frac{\partial^2 X^{j-1}}{\partial x^2} - \rho^j h^j \frac{\partial^2 u_0^j}{\partial t^2} + X^j - X^{j-1} = 0 \qquad (19)$$

$$D_1^j \frac{\partial u_0^j}{\partial x} + D_2^j w^j + D_3^j \frac{\partial \varphi^j}{\partial x} + D_4^j \frac{\partial X^j}{\partial x} + D_5^j \frac{\partial X^{j-1}}{\partial x} + \rho^j h^j \frac{\partial^2 w^j}{\partial t^2} + Z^j - Z^{j-1} = 0 \qquad (20)$$

$$F_1^j \frac{\partial^2 u_0^j}{\partial x^2} + F_2^j \frac{\partial^3 w^j}{\partial x^3} + F_3^j \varphi^j + F_4^j \frac{\partial^2 \varphi^j}{\partial x^2} + F_5^j X^j + F_6^j X^{j-1} + F_7^j \frac{\partial^2 X^j}{\partial x^2} + F_8^j \frac{\partial^2 X^{j-1}}{\partial x^2} = -\frac{\rho^j (h^j)^3}{12}\frac{\partial^3 w^j}{\partial t^2 \partial x} \qquad (21)$$

The coefficients C_k^j, D_k^j, and F_k^j are constants whose values depend on the elastic properties of the material and the dimensions of the shell structure. They are

$$C_1^j = B_{11}^j h^j; \quad C_2^j = (h^j/R^j) B_{12}^j$$

$$C_3^j = -[B_{11}^j (h^j)^3/12R^j]; \quad C_4^j = B_{11}^j[g_1^j + (1/R^j) g_4^j]$$

$$C_5^j = B_{11}^j[g_2^j + (1/R^j) g_5^j]; \quad C_6^j = B_{11}^j[g_3^j + (1/R^j) g_6^j]$$

$$D_1^j = C_2^j = (h^j/R^j) B_{12}^j; \quad D_2^j = (B_{22}^j/R^j)[(h^j/R^j) + C_1^j)]$$

$$D_3^j = (B_{12}^j/R^j) g_1^j - g_{31}^j;$$

$$D_4^j = (B_{12}^j/R^j) g_2^j - (h^j/2)[1 + (h^j/6R^j)] \qquad (22)$$

$$D_5^j = (B_{12}^j/R^j) g_3^j + (h^j/2)[1 - (h^j/6R^j)]$$

$$F_1^j = B_{11}^j (h^j)^3/12R^j = -C_3^j; \quad F_2^j = -B_{11}^j (h^j)^3/12 = C_3^j R^j$$

$$F_3^j = -g_{31}^j; \quad F_4^j = B_{11}^j g_7^j$$

$$F_5^j = -(h^j/2)[1 + (h^j/6R^j)]; \quad F_6^j = (h^j/2)[1 - (h^j/6R^j)]$$

$$F_7^j = B_{11}^j g_8^j; \quad F_8^j = B_{11}^j g_9^j$$

where the superscript j stands for the jth layer of a layered shell. The boundary conditions for rotationally symmetric deformation of cylindrical shell layer of orthotropic materials reduced directly from Eqs. (18) are:

1) Simply supported edge:

$$u_0^j (\text{or } N_x^j) = w^j = M_x^j = 0 \qquad (23a)$$

2) Fully fixed edge:

$$u_0^j = w^j = \partial w^j/\partial x = 0 \qquad (23b)$$

3) Free edge:

$$N_x^j = Q_x^j = M_x^j = 0 \qquad (23c)$$

For a shell having N layers, a total of N sets of differential equations together with N sets of boundary conditions along the edges and $N-1$ sets of continuity conditions needs to be solved. The continuity conditions between two adjacent jth and $j+1$th layers for the general case are

$$w^j = w^{j+1} \tag{24a}$$

$$u^j|_{z^j=h^j/2} = u^{j+1}|_{z^{j+1}=-h^{j+1}/2} \tag{24b}$$

$$v^j|_{z^j=h^j/2} = v^{j+1}|_{z^{j+1}=-h^{j+1}/2} \tag{24c}$$

A system of equations governing the behavior of multiple-layered cylindrical shells has thus been derived according to the theory, without using the classical Kirchhoff hypotheses. The rotary inertias are included in the general derivation. However, since all layers are considered to be thin, the corresponding mass moment of inertia may be neglected in the numerical computations. If the total thickness of multiple layers becomes large, it is conceivable that there could be a contribution from this term. However, Greenspon's work[19] for an isotropic material suggests that even in this case this is probably a higher-order effect.

References

[1] Shtaerman, I. I., "On the Theory of Symmetrical Deformation of Anisotropic Elastic Shells," *Izvestiya Kievskogo i Selsko-Khoziaistvennyi Institute*, 1924.

[2] Ambartsumian, S. A., *Theory of Anisotropic Shells*, Technical Translation F-118, May 1964, NASA.

[3] Dong, S. B., "Membrane Stresses in Laminated Anisotropic Shells," *Journal of Engineering Mechanics*, June 1964, pp. 53–69.

[4] Mushtari, K. M., "Some Generalizations on the Theory of Thin Shells with Application to the Solution of Stability of Elastic Anisotropic Shells," *Izvestiya Kazanskogo Fiziko-Matematicheskila Obshchestva*, Vol. 11, 1938.

[5] Mushtari, K. M., "Some Generalizations on the Theory of Thin Shells," *Prikladnaya Mat. Mekh.*, Vol. 2, No. 4, 1939, pp. 439–456.

[6] Ambartsumian, S. A., "On the Problem of the Calculation of Laminated Anisotropic Shells," *Izvestiia Akademii Nauk, Armyanskoi, S.S.R.*, Vol. 6, No. 3, 1953, pp. 15–35.

[7] Eason, G., "On the Vibration of Anisotropic Cylinders and Spheres," *Applied Scientific Research, Sec. A*, Vol. 12, 1963, pp. 81–85.

[8] Das, Y. C., "Vibrations of Orthotropic Cylindrical Shells," *Applied Scientific Research, Sec. A*, Vol. 12, 1964, pp. 317–327.

[9] Mirsky, I., "Three-Dimensional and Shell Theory Analysis for Axisymmetric Vibration of Orthotropic Shells," *Journal of Acoustic Society of American*, Vol. 39, No. 3, March 1966, pp. 549–555.

[10] Mirsky, I., "Axisymmetric Vibrations of Orthotropic Cylinders," *Journal of the Acoustical Society of America*, Vol. 35, No. 11, Nov. 1964, pp. 2106–2112.

[11] Kalnins, A., "On Free and Force Vibration of Rotationally Symmetric Layered Shells," *Journal of Applied Mechanics*, Vol. 32, No. 4, Dec. 1965, pp. 941–943.

[12] Ahmed, N., "Axisymmetric Plane-Strain Vibrations of a Thick Layered Orthotropic Cylindrical Shells," *Journal of the Acoustical Society of America*, Vol. 40, No. 6, Dec. 1966, pp. 1509–1516.

[13] Bert, C. W., Baker, J. L., and Egle, D. M., "Free Vibrations of Multilayer Anisotropic Cylindrical Shells," *Journal of Composite Materials*, Vol. 3, 1969, pp. 480–499.

[14] Ambartsumian, S. A., "Contributions to the Theory of Anisotropic Layered Shells," *Journal of Applied Mechanics Reviews*, Vol. 15, No. 4, April 1962, pp. 245–249.

[15] Boal, J. L. and Reissner, E., "Three-Dimensional Theory of Elastic Plates with Transverse Inextensibility," *Journal of Mathematics and Physics*, Vol. XXXIX, 1960, pp. 161–181.

[16] Novozhilov, V. V., *Thin Shell Theory*, 2nd ed., P. Noordhoff Ltd., Groningen-The Netherlands, 1964.

[17] Kraus, Harry, *Thin Elastic Shells*, Wiley, New York, 1967.

[18] Hsu, T., "Rotationally Symmetric Vibrations of Orthotropic Layered Cylindrical Shells," Ph.D. thesis, May 1969, Georgia Institute of Technology.

[19] Greenspon, J. E., "Vibrations of a Thick-Walled Cylindrical Shell—Comparison of the Exact Theory with Approximate Theories," *Journal of the Acoustical Society of America*, Vol. 32, No. 5, May 1960, pp. 571–578.